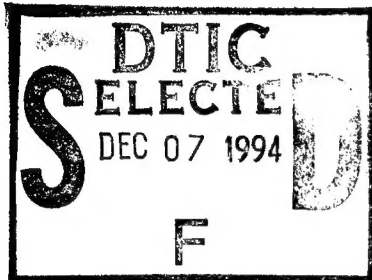NPS-OR-94-012

NAVAL POSTGRADUATE SCHOOL
Monterey, California

THE ROLES OF COUNTERFORCE AND ACTIVE DEFENSE IN COUNTERING THEATER BALLISTIC MISSILES

Kneale T. Marshall

September 1994

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ABSTRACT

This report contains the formulation and analysis of a model to measure, compare, and contrast the effects of counterforce (pre-launch attack) and active defense (post-launch attack) against tactical ballistic missiles (TBM's). It is shown that without counterforce an active defense system could require an impractical number of weapons to counter incoming missiles and/or their warheads. This number is shown to decrease geometrically as effective counterforce is used, so that the expected number of warheads killed increases dramatically with counterforce that is only modestly effective. Actual distributions of warheads reaching the target area are shown to be complex mixtures of binomial distributions. It is shown that normal approximations to these distributions based on the easily-calculated means and variances often agree poorly with the actual distributions. This is especially true when using effective counterforce.

1 Introduction

The purpose of this paper is to present and analyze a model of theater ballistic missile (TBM) launcher and missile flight operations so that comparisons can be made of the effectiveness of various strategies to counter the threat. The model presented here extends earlier analysis and results found in Conner, Ehlers, and Marshall [1993].

Figure 1 shows a schematic of the operations of a TBM launcher and the missile assumed in this report. Launchers are expected to be stored in some fixed storage area. When hostilities are about
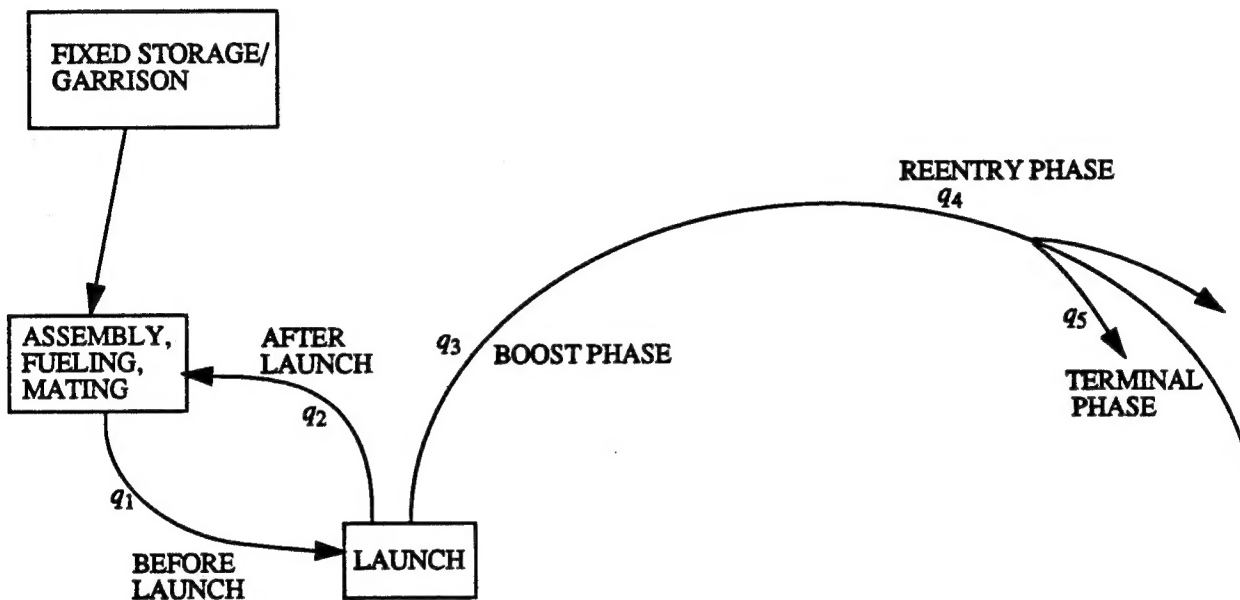
Figure 1: Schematic of Theater Ballistic Missile Operations

to commence the launchers will move to a forward area for assembly, fueling and mating with the missiles. From there a launcher will move to its launch area, and after launch will return to the forward area to prepare for the next launch. We assume that each launcher has the potential to launch m missiles, after which it must be taken out of service for an extended time. The reason could be that it must undergo extensive repair and refit, or it could run out of missiles. We also assume that each missile has n (≥ 1) warheads.

In this paper we assume that there are five phase in the TBM operation when the missile system could be attacked. These are

(a) Counterforce

    1. Attack of the launcher with mated missile before launch between assembly area and launch site.[1]

2. Attack of the launcher after missile launch either at the launch site or on return to assembly area.

(b) Active Defense

3. Attack of the missile during the boost phase,

4. Attack the missile on reentry before multiple warheads separate,

5. Attack each warhead in the terminal phase.

The effectiveness of attacking the system in each of these five phases is assume to be summarized by a kill probability p_i for the i-th phase, or equivalently by a survival probability q_i, where $q_i = 1 - p_i$. Although it is more usual to formulate a model in terms of kill probabilities, survival probabilities are used in Section 2 because of the simplification that results in model development and presentation of results. Our objectives are to find the probability distribution, mean, and variance of the number of warheads reaching the target area from each launcher, and the expected number of weapons required in each phase, in terms of the maximum number missiles per launcher (m), the number of warheads per missile (n) and the five survival probabilities for the five phases as shown in Figure 1. Using expressions for these quantities, in Section 3 we compare the effect of changing the model parameters to demonstrate that counterforce, with effectiveness measured by q_1 and q_2, will almost surely be a necessary part of a layered defense system; without at least a modest success rate in prosecuting the launchers effective active defense may not be feasible.

2 The Anti-TBM Model

We build the mathematical model in stages following the missile's path from being mounted on the launcher to its or its launcher's destruction, or the arrival of its warheads in the target area. First we develop the probability distribution, mean and variance of the number of missiles that are successfully launched from a given launcher. These clearly will depend on the counterforce effort against the launcher. Next we derive the probability distribution, mean and variance of the number of missiles that survive the boost and reentry phases. Finally we find expressions for the probability distribution, mean and variance of the number of warheads that survive the final phase. The distribution of the warheads surviving to reach the target area is a complex mixture of Binomial probabilities. The section ends with numerical examples to illustrate the results. A detailed analysis using the model is presented in Section 3.

2.1 Launcher Movement Phases

Let X be the number of missiles launched from a given launcher before it is either destroyed or has launched m missiles. We assume independent attacks each time the launcher attempts an outward journey to the launch site, and similarly for each time it attempts an return journey to reload. Thus X is a random variable that can take on any integer value from 0 (the launcher is destroyed on the first outward journey) to m (all attempts to destroy the launcher fail). Note that $X > i$ if and only if the launcher survives the first outward journey, and then survives i succeeding cycles back to the reload point and out again to the launch site. Thus

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[1.] It is understood that a launcher may employ a number of tactics on its way to or from the launch site, such as stopping in hide sites. The model summarizes the effects of these strategies in a single survival or kill probability.

$$Pr\{X > 0\} = q_1$$
$$Pr\{X > 1\} = q_1(q_1 q_2)$$
$$Pr\{X > 2\} = q_1(q_1 q_2)^2$$
$$\ldots$$
$$Pr\{X > m\text{-}1\} = q_1(q_1 q_2)^{m\text{-}1}$$
$$Pr\{X > m\} = 0.$$

By summing this cumulative tail distribution (see result 1 of the Appendix):

$$E[X] = \sum_{i=1}^{m-1} q_1(q_1 q_2)^i = \frac{q_1(1 - (q_1 q_2)^m)}{(1 - q_1 q_2)} . \tag{1}$$

This equation holds if both $0 \le q_1 < 1$ and $0 \le q_2 < 1$, and is equal to m when both q_1 and q_2 are equal to 1 (zero effect in killing the launcher before or after launch).

To find its variance we need to find its second moment. Using result 2 of Appendix, the second moment of X is

$$E[X^2] = \frac{2q_1(q_1 q_2)[1 - m(q_1 q_2)^{m-1} + (m-1)(q_1 q_2)^m]}{(1 - (q_1 q_2))^2} + \frac{q_1(1 - (q_1 q_2)^m)}{1 - (q_1 q_2)} \tag{2}$$

when both $0 \le q_1 < 1$ and $0 \le q_2 < 1$, and is equal to m^2 when both q_1 and q_2 are equal to 1.

We find the variance of X in the usual way by subtracting the square of Equation (1) from Equation (2).

We now turn to finding the expected number of weapons required in the first two phases. Before attempting to do this it is necessary to make two important assumptions which are assumed to hold in all five phases. First, we assume that every time there is an opportunity to attack the launcher, the missile, or one of its warheads, this opportunity is taken and prosecuted with a single weapon. It may be that in practise more than one weapon is used, so that the numbers determined by the model in this report can be thought of as lower bounds. Second, the extreme case of some kill probability being zero in a given phase can be obtained in one of two ways, either (i) by not attempting an attack during that phase, or (ii) by attacking with a completely ineffective weapon system. In this paper we assume that the first of these is true; any time we use a p_i of zero (q_i of one) in phase i we assume no weapons are expended in phase i. The expected numbers of weapons required should not be interpreted as estimates of weapons requirements in actual operations. In this paper they are intended as an aid in gaining insight when comparing the effectiveness of changing kill probabilities in the various phases.

Let W_{BL} and W_{AL} be the numbers of weapons used in the "before launch" and "after launch" phases respectively against the launcher. Notice that if no missiles are launched, W_{AL} is zero (the launcher was destroyed on its first outward journey). It is easy to show that no matter how many

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missiles are launched from a given launcher, $W_{AL} = X$ and its first two moments are given by Equations (1) and (2).

By following the cycle of the launcher one can see that the cumulative tail distribution of W_{BL} is given by

$$Pr\{W_{BL} > i\} = (q_1 q_2)^i \qquad \text{if } i = 0, 2, 4, \dots, (m\text{-}1),$$
$$= 0 \qquad \text{if } i \geq m.$$

Using result 1 of the Appendix,

$$E[W_{BL}] = \frac{1 - (q_1 q_2)^m}{1 - q_1 q_2},$$

and by comparing this with Equation (1) we can see that

$$E[W_{BL}] = E[X]/q_1. \tag{3}$$

As our analysis progresses through the boost and reentry phases, expressions are found that require the probability mass function (pmf) of X. From the cumulative tail distribution above this is seen to be

$$p_X(0) = 1 - q_1,$$

$$p_X(i) = q_1(1 - q_1 q_2)(q_1 q_2)^{i-1} \quad , i = 1, 2, \dots, m\text{-}1, \tag{4}$$

$$p_X(m) = q_1(q_1 q_2)^{m-1}.$$

2.2 The Boost and Reentry Phases

The boost phase and reentry phase survival probabilities are q_3 and q_4 respectively (see Figure 1). Let the number of missiles surviving both of these phases (per launcher) be Y. Clearly this is also a random variable, and if we assume that the attempt to shoot down a given missile in either phase is independent of the outcomes of earlier or later attempts at other missiles, the conditional random variable $[Y|X]$ has a Binomial distribution with parameters X and $q_3 q_4$. Thus $E[Y|X] = X q_3 q_4$ and $Var[Y|X] = X q_3 q_4 (1 - q_3 q_4)$. By unconditioning on X, the expected number of warheads surviving the through the reentry phase is

$$E[Y] = q_3 q_4 E[X] \tag{5}$$

where $E[X]$ is given by Equation (1).

The variance of Y is found using the standard conditional variance argument,

$$Var[Y] = E_X[Var[Y|X]] + Var_X[E[Y|X]],$$

so

$$Var[Y] = q_3 q_4 (1 - q_3 q_4) E[X] + (q_3 q_4)^2 Var[X], \tag{6}$$

where Equations (1) and (2) are used to find $Var[X]$.

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To find the pmf of Y, note that

$$p_{Y|X}(j|i) = b(j; i, q_3q_4)$$

where $0 \le j \le i, 0 \le q_3q_4 \le 1$, and $b(j; i, p) = \binom{i}{j} p^j (1-p)^{i-j}$.

Unconditioning on X we find

$$p_Y(j) = \sum_{i=j}^{m} b(i; j, q_3q_4) p_X(i), \quad j = 0, 1, 2, \ldots, m, \tag{7}$$

where the $p_X(i)$'s are given in Equation (4).

Let W_B and W_R be the number of weapons used in the boost and reentry phases respectively against the missiles from a given launcher, and assume that exactly one weapon is used against each in each phase. If X survive launch, $W_B = X$ and W_R is a Binomial random variable with parameters X and q_3. Thus $E[W_B] = E[X]$, and $E[W_R] = q_3E[X]$.

2.3 The Final Phase

In the final phase the probability that a given warhead survives an attack is q_5. Again we assume independence among all attempts to destroy incoming warheads. Let the number of warheads surviving the final phase from the i-th incoming missile be Z_i, $i = 1, 2, \ldots, Y$. Each Z_i is a Binomial random variable with parameters n and q_5, so $E[Z_i] = nq_5$ and $Var[Z_i] = nq_5(1-q_5)$. Let the number of warheads surviving the final phase (per launcher) be H, so

$$H = \sum_{i=1}^{Y} Z_i.$$

Conditioning on Y, $E[H|Y] = nYq_5$ and $Var[H|Y] = YVar[Z_i] = nYq_5(1-q_5)$. Unconditioning,

$$E[H] = nq_5E[Y] \tag{8}$$

and

$$Var[H] = nq_5(1-q_5)E[Y] + n^2q_5^2Var[Y], \tag{9}$$

where $E[Y]$ and $Var[Y]$ are given by Equations (5) and (6) respectively.

The pmf of H, $p_H(k)$, is found in a similar way by first conditioning on Y. If $Y = 0$ (no missiles survive through the reentry phase) no warheads can reach the target area, so $p_{H|Y}(0 | 0) = 1$. If $Y = j > 0$, H is the sum of j identically distributed binomials so that $p_{H|Y}(k | j) = b(k; nj, q_5)$. Unconditioning,

$$p_H(k) = \sum_{j=k}^{m} b(k; nj, q_5) p_Y(j), \quad k = 0, 1, 2, \ldots, mn, \tag{10}$$

where the $p_Y(j)$'s are given in Equation (7).

Let W_F be the number of weapons used in the final phase. If Y missiles survive the reentry phase and each carries n warheads, then $W_F = nY$. Thus the results on Y can be used to calculate the measure of interest on W_F.

Figure 2 demonstrates the model by showing the cumulative tail distribution of H for three different sets of survival probabilities. For all three cases the number of missiles per launcher (m) is 20, and the number of warheads per missile (n) is 10. The rightmost curve is obtained using no (or completely ineffective) counter force ($q_1 = q_2 = 1$), boost and reentry survival probabilities (q_3 and q_4) of 0.7, and a final phase warhead survival probability (q_5) of 0.4. The center curve is obtained by decreasing $q3$ and q_4 from 0.7 to 0.6, and q_5 from 0.4 to 0.3. The leftmost curve is obtained using the original set of parameters but decreasing both q_1 and q_2 from 1 to 0.9. Clearly a modest increase in kill probability in counterforce operations from 0 to 0.1 has a dramatic effect on the number of warheads reaching the target area. An increase in kill probability from 0 to 0.1 in the two phases of the launcher shows a drop in the 10-th percentile from 52 warheads to 23, compared to a drop from 53 to 31 for a similar increase in kill probability in the boost, reentry and final phases. Another way to interpret the three curves is to note that the chance of *at most* 20 warheads (10% of a potential of 200) reaching the target area is 3% for the base case. With a given improvement in active defense this increases to 46%, but if that improvement were made in counterforce instead of active defense it would increase to 87%. These numbers are shown in Column 2 of Table 1. Columns 3 through 6 show the expected number of weapons used in each phase. A small improvement in counterforce effectiveness sharply decreases the number of weapons required for active defense. Note that the zero entries in columns 3 and 4 result from the assumption that when $q_1 = q_2 = 1$, it is assumed that no counterforce is attempted.

The next section contains a more detailed analysis of the model as parameter values are varied.

3 Model Analysis

Throughout this section results are demonstrated using kill probabilities rather than survival probabilities p_1 through p_5, where $p_i = 1 - q_i$. We refer to a kill probability vector which is defined to be $(p_1, p_2, p_3, p_4, p_5)$. For example $(0, 0.2, 0.3, 0.5, 0.6)$ represents no chance of killing the launcher in its outward journey to the launch site, a 20% chance of kill on its return journey to reload, a 30% chance of killing the missile in its boost phase, a 50% chance in its reentry phase, and a 60% chance of killing each warhead in the final phase.

Theater anti-missile defense today consists primarily of the use of the PATRIOT system in the final phase. The navy Aegis ship anti-missile defense system is currently being considered for modification for the reentry phase of anti-TBM mission, and the army is developing the THAAD (theater high altitude air defense) system for this same phase. The air force is currently developing boost phase systems. Although some work has been done on detecting and destroying launchers prior to or after a launch, operational experience in Desert Storm showed that current systems and
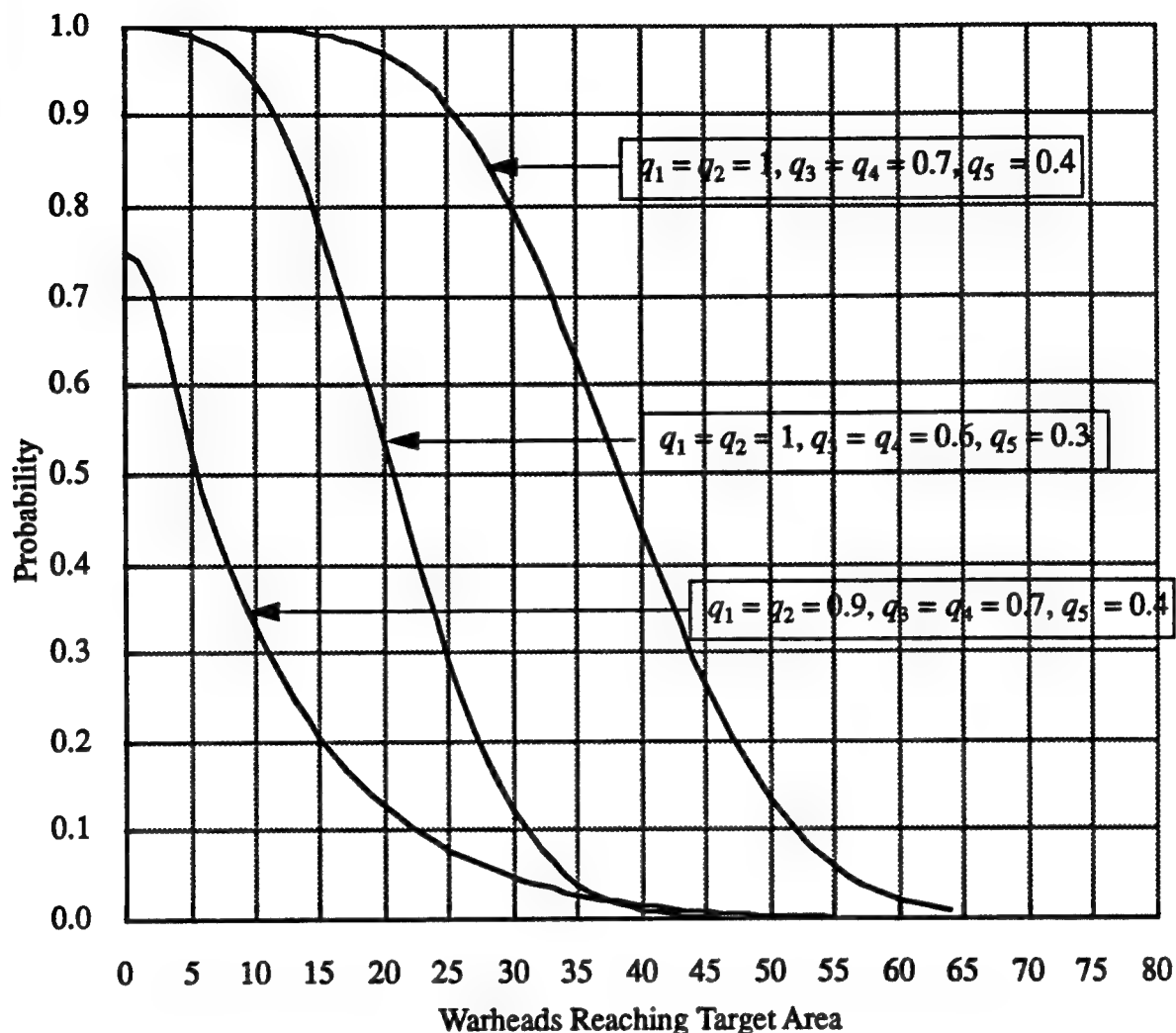
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Figure 2: Cumulative Tail Distributions of Warheads Reaching Target Area

Case	$Pr\{H \leq 20\}$	$E[W_{BL}]$	$E[W_{AL}]$	$E[W_B]$	$E[W_R]$	$E[W_F]$
$q_1 = q_2 = 1, q_3 = q_4 = 0.7, q_5 = 0.4$	0.03	0	0	20	14	98
$q_1 = q_2 = 1, q_3 = q_4 = 0.6, q_5 = 0.3$	0.46	0	0	20	12	72
$q_1 = q_2 = 0.9, q_3 = q_4 = 0.7, q_5 = 0.4$	0.87	5.2	4.7	4.7	3.3	23

Table 1: Sample Output for Numerical Example

operational doctrine are ineffective. This current state can be modeled by setting p_1, p_2, p_3 and p_4 all equal to 0. We can set p_5 at some value depending on how well one believes the PATRIOT works. As a base case by which to measure possible system improvement we set p_5 to 0.7. Thus

$$\text{Base Case Kill Probability Vector} = (0, 0, 0, 0, 0.7). \qquad (11)$$

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Also as a base case we assume that a launcher can launch at most 20 missiles before requiring major overhaul, or before it runs out of missiles, so $m = 20$.

We look at three measures of effectiveness for the (random) number of warheads arriving in the target area, H. These are (i) the mean $E[H]$, (ii) the median, or that value h such that $Pr\{H \leq h\} = 0.5$, and (iii) the ninetieth percentile, or that value h such that $Pr\{H \leq h\} = 0.90$. We also look at the expected number of active defense weapons required ($E[W_B]$, $E[W_R]$, and $E[W_F]$), and the expected number of counterforce weapons ($E[W_C]$). We first look at today's case where there is only one warhead per missile ($n = 1$), and show how some performance measures are affected by improving kill probabilities in each of the first four phases. This is followed by a similar analysis when multiple warheads are considered.

3.1 Single Warhead Analysis

The mean numbers of warheads (and hence missiles since we are assuming one warhead per missile) that arrive in the target area shown plotted in Figure 3 as a function of the kill probability at a particular stage. The figure contains three curves. All three start at the same point (0,6) because the expected number of warheads reaching the target area, $E[H]$, is 6 when $m = 20$, $n = 1$, the base case probabilities are given in (11), and Equations (1), (5), and (8) are used. We investigate the ef-
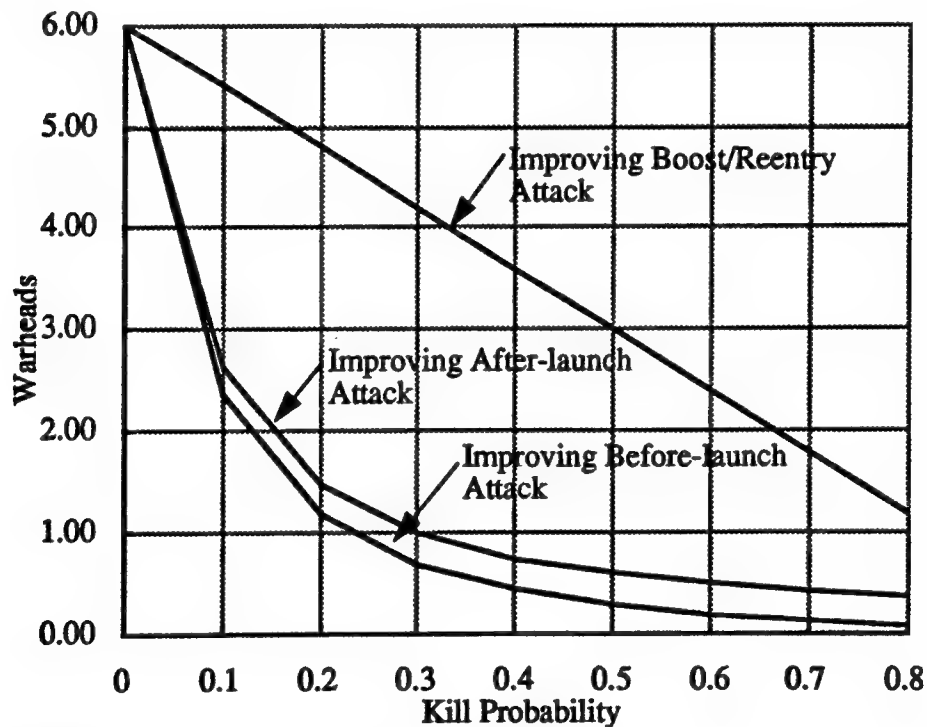

Figure 3:Mean Number of Warheads Reaching Target

fect on $E[H]$ of increasing each of the four zero kill probabilities in (11) one at a time.

The upper curve is found by increasing the kill probability of either the boost (p_3) or reentry (p_4) phase from its base value of 0 up to 0.8. In either case it decreases linearly with a slope of -6. The middle and lower curves are obtained by increasing p_2 and p_1 respectively over the same range. The difference in the effect of a small increase in kill probability in the counterforce phases when compared to the active defense stages is dramatic; an increase from 0 to 0.1 in either to boost or reentry phases reduces $E[H]$ from 6 to 5.4, whereas this same increase in the either of the counterforce stages reduces it from 6 to approximately 2.5. This significant improvement is caused by the fact that once a launcher (and its crew) is destroyed it can no longer fire missiles, causing a geometric reduction in $E[H]$. In the active defense stages a kill results in the destruction of only one missile. The small improvement in increasing p_1 rather than p_2 is caused by the fact that keeping p_1 at zero means the first missile from a launcher will be launched for certain, whereas increasing p_1 gives a chance to destroy the launcher before its first missile flies.

Figure 4 contains a similar analysis using the median number of warheads reaching the target area rather than the mean. Similar results are found. For the base case the median of H is 5.4. Increasing the boost or reentry kill probabilities from 0 to 0.1 reduces this to 4.8, whereas this increase in p_1 or p_2 reduces it to 1.3 and 1.6 respectively. In other words, using a kill probability vector (0.1, 0, 0, 0, 0.7) there is a fifty percent chance that fewer than 1.3 warheads will reach the target area, whereas using (0, 0, 0, 0.1, 0.7) or (0, 0, 0.1, 0, 0.7) this number is 4.8.
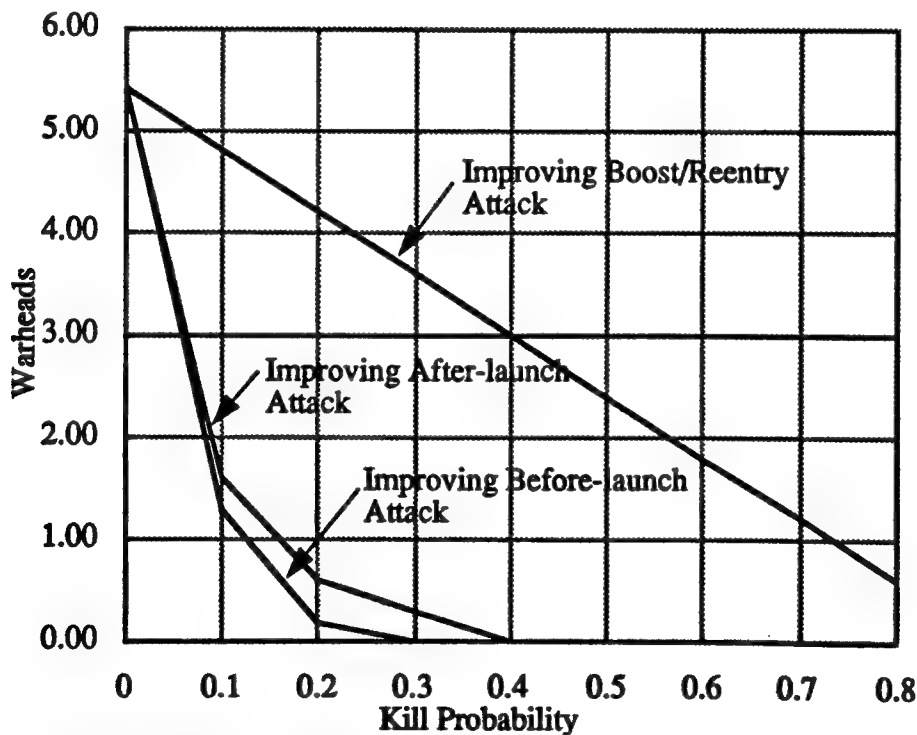

Figure 4: Median Number of Warheads Reaching Target

Figure 5 contains a similar analysis using the ninetieth percentile of the number of warheads reaching the target. For the base case there is a ninety percent chance that the number of warheads reaching the target area from a given launcher is no more than 8.2. Increasing the boost or reentry
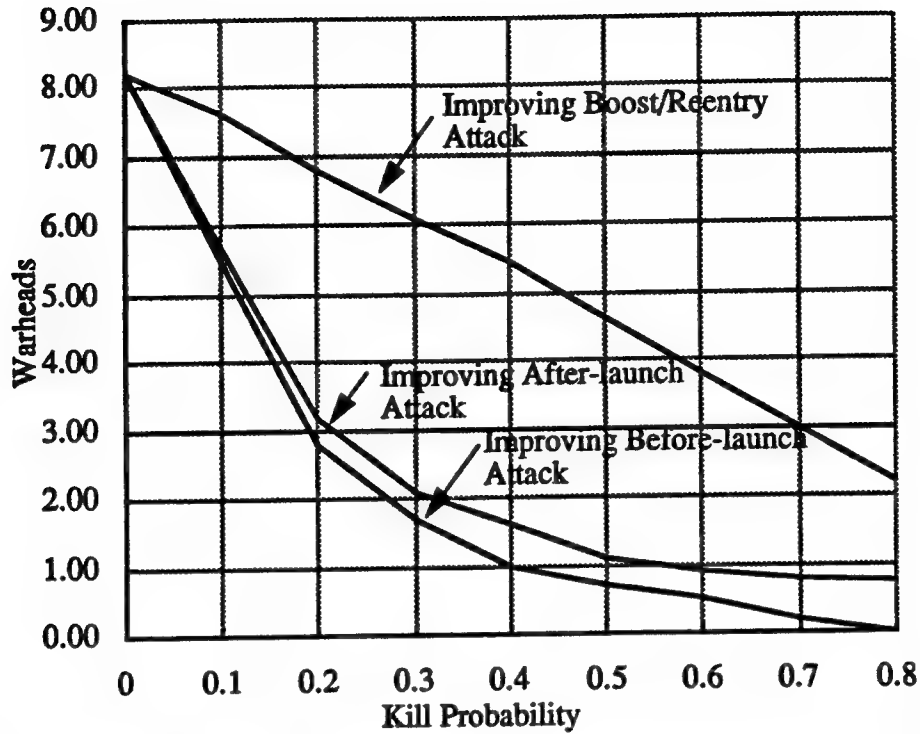
Figure 5: Ninetieth Percentile of the Number of Warheads Reaching Target

kill probabilities from 0 to 0.1 reduces this to 7.6 whereas an increase from 0 to 0.1 in p_1 or p_2 reduces it to 5.5 or 5.7 respectively. Although by using this measure of effectiveness there is less of a difference between improving counterforce and active defense, the difference is still significant.

We now turn to measuring the effects of changing kill probabilities on the expected numbers of weapons used in each phase. Starting from the base case we assume that a zero kill probability in a given phase indicates that no attempt is being made to kill the launcher or missile in that phase. Table 2 demonstrates typical results that can be obtained from the model. For the base case the

Kill Probability Vector	$E[W_{BL}]$	$E[W_{AL}]$	$E[W_B]$	$E[W_R]$	$E[W_F]$	Expected Warheads Killed/Weapon
(0, 0, 0, 0, 0.7)-Base Case	0	0	0	0	20	0.70
(0, 0, 0, 0.2, 0.7)	0	0	0	20	16	0.42
(0, 0, 0.2, 0, 0.7)	0	0	20	0	16	0.42
(0, 0.2, 0, 0, 0.7)	0	4.94	0	0	4.94	1.88
(0.2, 0, 0, 0, 0.7)	4.94	0	0	0	3.95	2.11

Table 2: Effect of Increasing Kill Probabilities on Weapons Numbers and Effectiveness

expected number of weapons used per launcher when no attempt is made to destroy the missile before the final phase, and assuming one weapon for each warhead, is equal to the number of missiles time warheads per missile that a launcher can launch. In this example that is 20. Also for the base case the expected number of warhead kills per weapon is equal to the final phase kill probability as should be expected. The remaining rows in Table 2 show the effect of increase the kill probability of each phase in turn from 0 to 0.2. Note the dramatic drop in the requirement for weapons in the final phase by having a modest effectiveness in counterforce versus the same effectiveness in the boost or reentry phases. In those phases a modest kill probability significantly *increases* the warhead kills/weapons used ratio.

3.2 Multiple Warhead Analysis

We repeat the analysis of Section 3.1 using the same base case kill probability vector shown in (11) and twenty missiles per launcher ($m = 20$), but in this section we assume each missile carries ten warheads ($n = 10$). The same types of results are illustrated in Figures 6, 7 and 8 as were seen in Figures 3, 4 and 5. In fact since the mean is linear in n the curves in Figure 5 are the same as
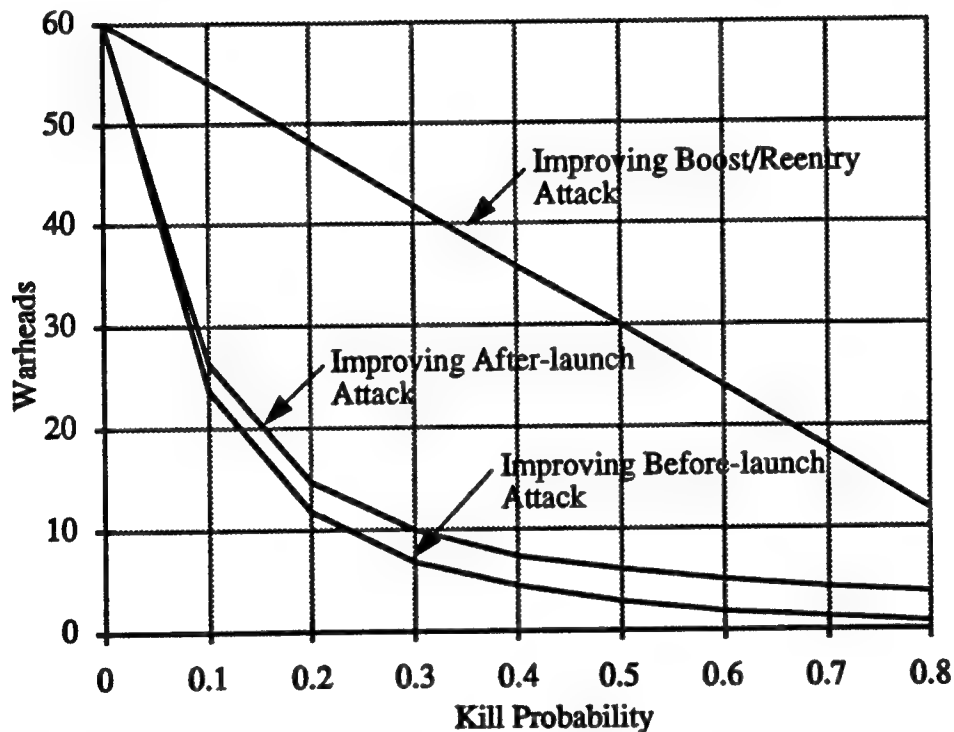

Figure 6: Mean Number of Warheads Reaching Target, Ten Warheads per Missile

those in Figure 3 except the vertical scale has changed by a factor of 10. There is no simple relationship between the median or the ninetieth percentile and n, although over some of the range of the kill probability the relationship appears to be approximately linear. For example, from Figure 4 with $n = 1$ we see that a median number 2 for H (90% kill of the twenty possible warheads) can be achieved if p_1 or p_2 are close to 0.08, whereas in the boost or reentry phases we would p_3 or p_4 to
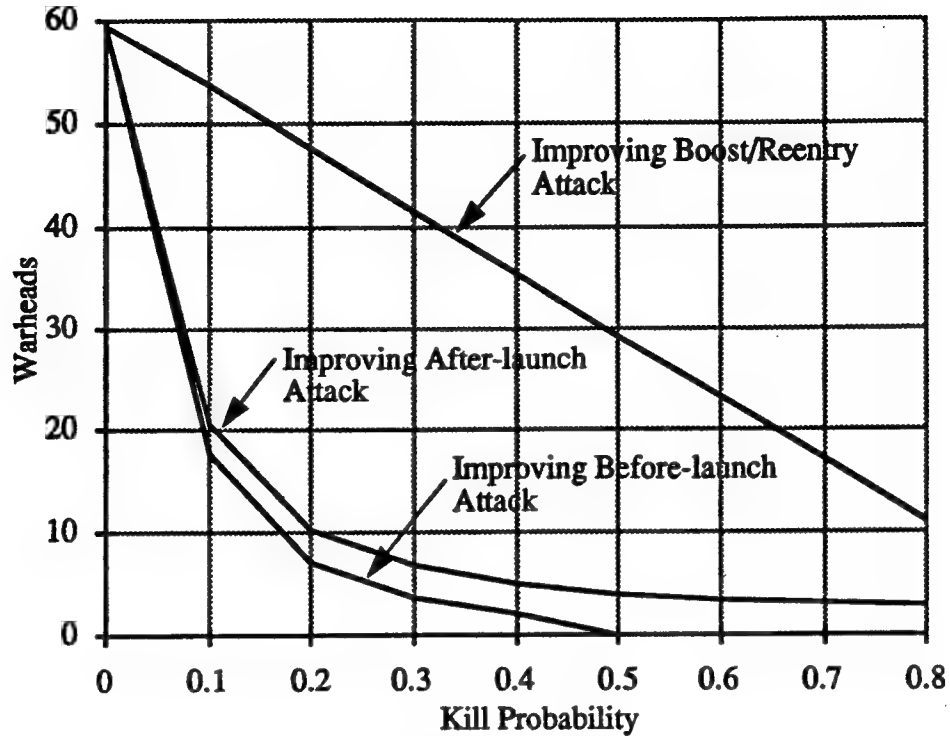
Figure 7: Median Number of Warheads Reaching Target, Ten Warheads per Missile

be 0.56 to achieve this success. From Figure 5 with $n = 10$ we see that a median number 20 for H (90% kill of the two hundred possible warheads) can be achieved if p_1 or p_2 are close to 0.1, whereas in the boost or reentry phases we would p_3 or p_4 to be 0.65. Similarly, from Figure 5 we see that to achieve a ninetieth percentile of 2 when $n = 1$ requires either a p_1 or p_2 of about 0.28 or a p_3 or p_4 of 0.81; from Figure 5 a ninetieth percentile of 20 when $n = 10$ requires either a p_1 or p_2 of about 0.30 or a p_3 or p_4 of 0.79.

Table 3 shows the expected numbers of weapons required at each stage and the expected warhead kills per weapon when $n = 10$. By comparing the results with those in Table 2 it is clear that the required expected numbers of weapons at the counterforce, boost, or reentry phases does not change when warheads per missile increase from 1 to 10, but the number of weapons in the final stage increases by a factor of ten. These results should be expected since a successful kill at any phase before the warheads separate is assume to kill all n warheads. Note that the expected number of warheads killed per weapon increases significantly as n increases the earlier one can attack the TBM operation. In other words, counterforce is increasingly effective as the number of warheads carried by the missile increases.

3.3 Normal Approximations

For given values of m, n, and a kill probability vector, it is easy to calculate the expected value of H using Equations (1), (5), and (8); likewise one can easily find the variance using Equations (1), (2), (5), (6), and (9). But to find percentiles such as the median or the ninetieth percentile re-
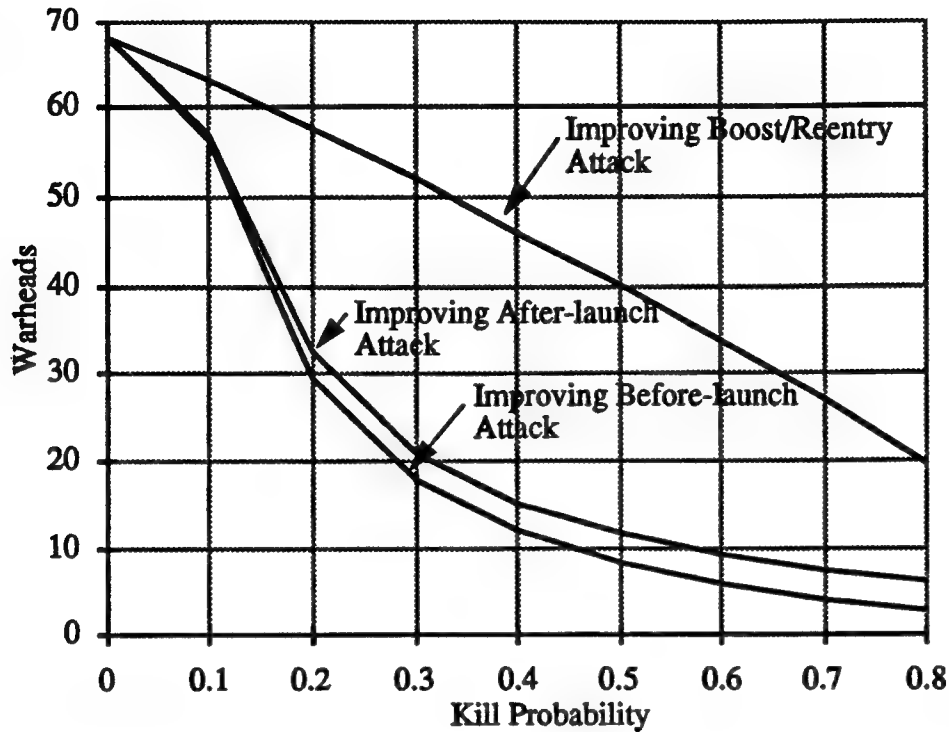
Figure 8: Ninetieth Percentile of Warheads Reaching Target, Ten Warheads per Missile

Kill Probability Vector	$E[W_{AL}]$	$E[W_{AL}]$	$E[W_B]$	$E[W_R]$	$E[W_F]$	Expected Warheads Killed/Weapon
(0, 0, 0, 0, 0.7)-Base Case	0	0	0	0	200	0.70
(0, 0, 0, 0.2, 0.7)	0	0	0	20	160	0.84
(0, 0, 0.2, 0, 0.7)	0	0	20	0	160	0.84
(0, 0.2, 0, 0, 0.7)	0	4.94	0	0	49.4	3.41
(0.2, 0, 0, 0, 0.7)	4.94	0	0	0	39.5	4.23

Table 3: Expected Weapons Numbers and Effectiveness with Ten Warheads per Missile

quires the distribution function of H, a much more complex calculation using Equations (4), (7), and (10). These equations were used to find the curves in Figures 2, 4, 5, 7, and 8. Recall that H is not a simple sum of independent random variables, but results from a complex set of five random events, the first two of which have a truncated geometric distribution, the next two a conditional binomial distribution, and the last is a random sum of these weighted binomials. Even so, one might suspect that its distribution is approximately normal for at least some range of the parameter values, in which case the percentiles can be estimated using only the mean and variance of H. We

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investigate the appropriateness of a normal approximation for the median and ninetieth percentiles of H in this section.

Since the normal is a symmetric distribution its mean and median are equal. Table 4 contains

Kill Probability Vector	Ten Warheads per Missile $(n = 10)$		One Warhead per Missile $(n = 1)$	
	Median	Normal Approximation	Median	Normal Approximation
$(0, 0, 0, 0, 0.7)$-Base Case	59.3	60.0	5.4	6.0
$(0, 0, 0, 0.2, 0.7)$ or $(0, 0, 0.2, 0, 0.7)$	47.6	48.0	4.2	4.8
$(0, 0.2, 0, 0, 0.7)$	10.3	14.8	0.6	1.5
$(0.2, 0, 0, 0, 0.7)$	7.1	11.9	0.2	1.2
$(0.2, 0.2, 0.3, 0.5, 0.7)$	NA	2.3	NA	0.2

Table 4: Normal Approximation for the Median

actual medians and normal approximations for the base case and kill probability vectors used in Sections 3.1 and 3.2, and an example that assumes positive kill probabilities in all five stages. The normal approximation seems to perform reasonably well for the ten warhead case when there are zero kill probabilities in the counterforce stages; it does less well in the single warhead case. When p_1 and/or p_2 are significantly larger than zero, the distribution of H is highly skewed and the normal approximation for the median is poor. The entries NA (not applicable) in the table indicate that the probability that H is zero is larger than 0.5 so that no median value exists.

Figure 9 contains cumulative tail distributions (solid lines) and normal approximations (dashed lines) for the kill probability vectors in Table 3 and one warhead per missile. For none of the examples is the normal approximation close to the actual distribution except in the extreme tails. It is particularly poor when there is a positive probability of kill by counterforce.

Figure 10 contains cumulative tail distributions (solid lines) and normal approximations (dashed lines) for the kill probability vectors in Table 3 and ten warheads per missile. When there is no counterforce the normal approximation is close to the actual distribution over the whole range, but again there are significant differences when there is a positive probability of kill by counterforce.

As one might expect the approximation does quite well when H is a fixed (non-random) sum of binomial random variables. Since this number is considerably larger when multiple warheads are present it does significantly better in this case. With positive counterforce probabilities the truncat-
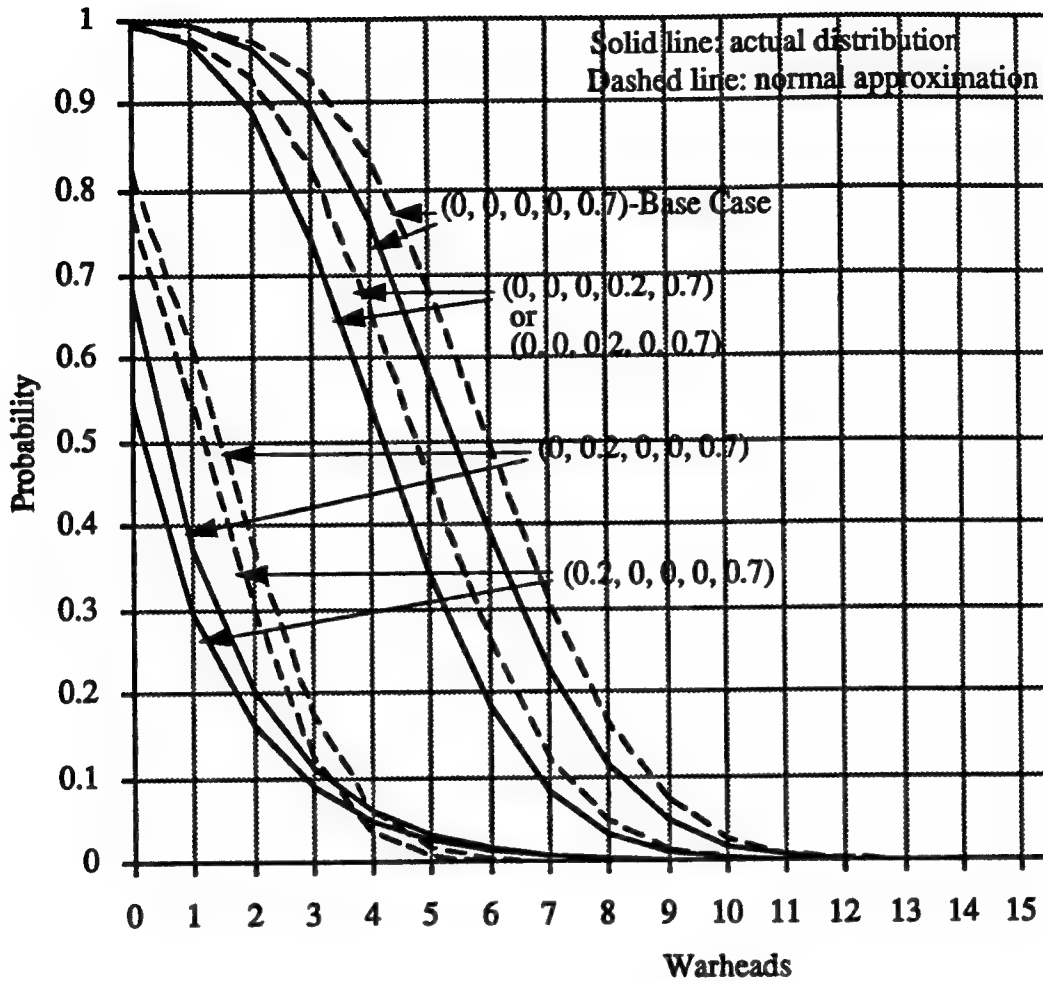
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Figure 9: Cumulative Tail Distributions and Normal Approximations, $n = 1$

·ed geometric distribution of the number of missiles launch leads to skewing of the distribution of H. In this case the normal approximation shows significant error.

It is not recommended that the normal approximation be used for the median (or equivalently that the median and mean be assumed to take on the same value). Nor is it recommended that it be used as an approximation to the tail distribution unless multiple warheads are assumed to be present and the only significant source of uncertainty is in the final stages of the TBM operation.
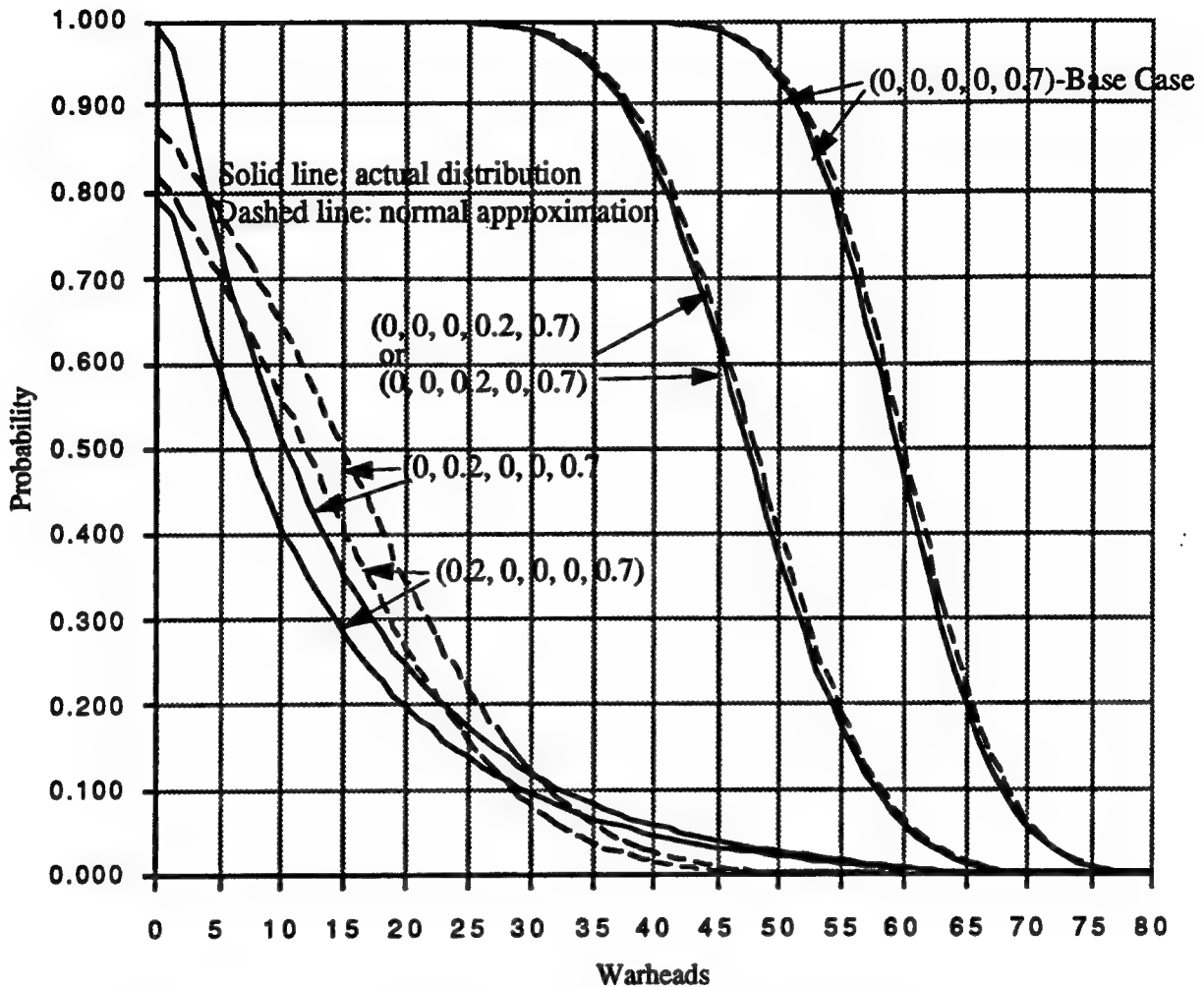
Figure 10: Cumulative Tail Distributions and Normal Approximations, $n = 10$

4 Conclusions

The model in this report shows that both counterforce and active defense will form essential parts of any future successful system for theater ballistic missile defense. Without counterforce it will be relatively easy for the enemy to overwhelm a feasible active defense system. A system that can successfully destroy launchers and their crews will provide considerable leverage in reducing the numbers of active defense weapons required; this leverage increases dramatically as the number of warheads on each missile increases. The model allows the calculation of percentiles of the numbers of warheads destroyed rather than simple expected values.

Past experience in finding and destroying launchers has demonstrated little success in this areas. As was discussed in Conner, Ehlers, and Marshall [1993], success will most likely require a far more structured approach than has been used. A model for such a structure is that used in anti-submarine warfare where great experience has been gained in the past fifty years at finding and de-

stroying torpedo underwater missile launchers. It is expected that the successful counterforce against launchers on land will require efforts in cuing, search, detection, localization, classification and destruction. Current efforts can be thought of as attempting to skip from cuing (for example, flaming datum information after launch) to attack. Future reports will consider how one might best accomplish the in-between phases to produce successful counterforce against mobile missile launchers.

References

1. Conner, George W., Ehlers, Mark A., and Marshall, Kneale T., "Countering Short Range Ballistic Missiles," Technical Report NPS-OR-93-009, Department of Operations Research, Naval Postgraduate School Monterey, CA 93940, January, 1993.

APPENDIX

Expressions are derived below for the first two of a non-negative integer-valued random variable in terms of its cumulative tail distribution. Let X be such a random variable and let $p_i = Pr\{X = i\}$, $i = 0, 1, 2, \ldots$.

First Moment

$$E[X] = p_1 + 2p_2 + 3p_3 + 4p_4 + \ldots$$

$$= p_1$$
$$+ p_2 \qquad + p_2$$
$$+ p_3 \qquad + p_3 \qquad + p_3$$
$$+ p_4 \qquad + p_4 \qquad + p_4 \qquad + p_4$$
$$+ \ldots \qquad\qquad\qquad\qquad\qquad\qquad \ldots$$

$$= Pr\{X>0\} + Pr\{X>1\} + Pr\{X>2\} + Pr\{X>3\} + \ldots$$

$$\text{Result 1:} \quad E[X] = \sum_{i=0}^{\infty} Pr\{X > i\}$$

Second Moment

$$E[X^2] = p_1 + 4p_2 + 9p_3 + 16p_4 + \ldots$$

$$= p_1$$
$$+ p_2 + p_2 + p_2 + p_2$$
$$+ p_3 + p_3 + p_3 + p_3 + p_3 + p_3 + p_3 + p_3 + p_3$$
$$+ p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4 + p_4$$
$$+ \ldots \qquad\qquad\qquad\qquad\qquad\qquad\qquad \ldots$$

$$= Pr\{X>0\} + 3Pr\{X>1\} \qquad + 5Pr\{X>2\} \qquad\qquad + 7Pr\{X>3\} + \ldots$$

$$\text{Result 2:} \quad E[X^2] = \sum_{i=0}^{\infty} (2i+1) Pr\{X > i\} = 2 \sum_{i=1}^{\infty} i Pr\{X > i\} + E[X]$$

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